UNDERGRADUATE TEXTS IN MATHEMATICAL COMICS

A TOUR OF COMPLEX ANALYSIS

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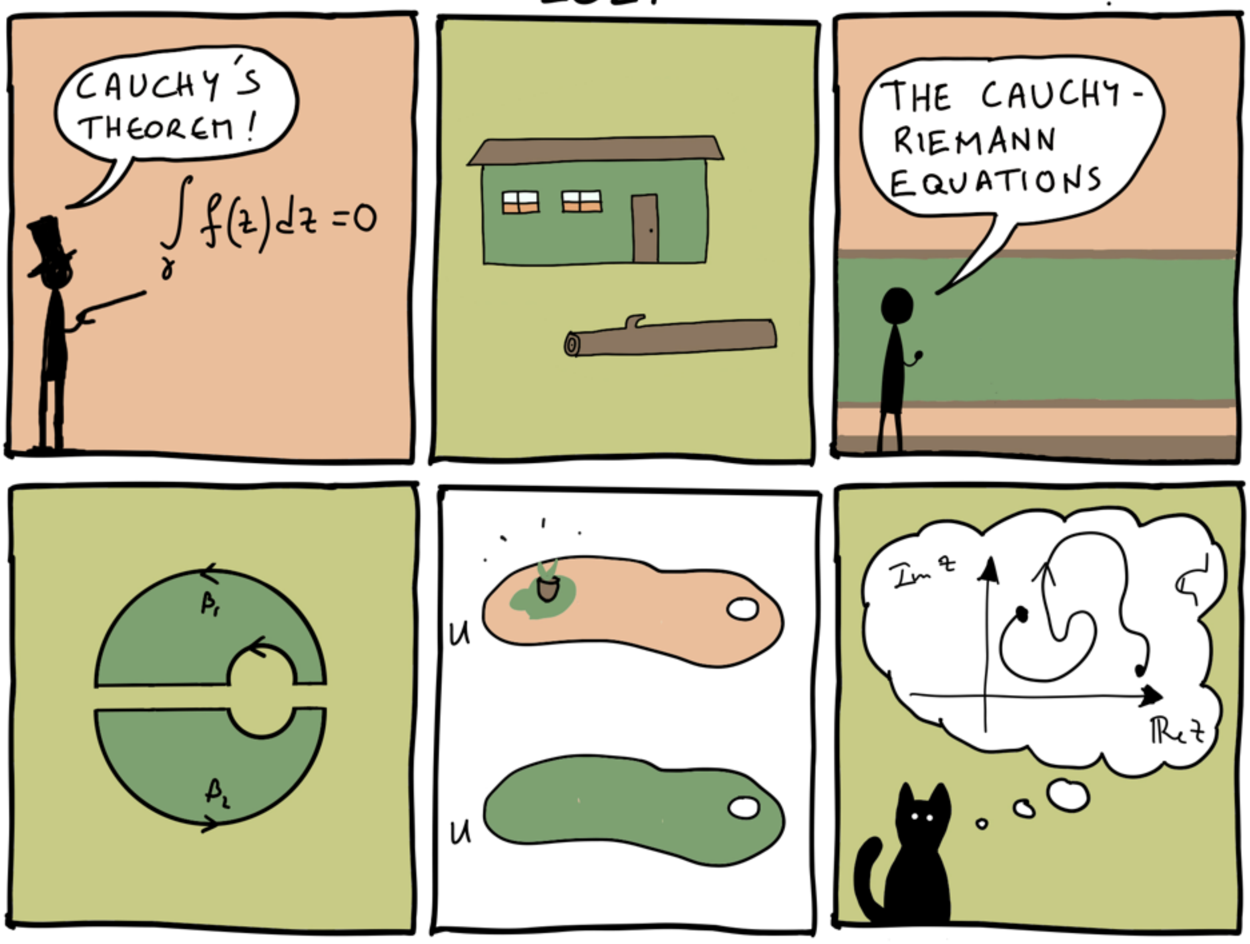
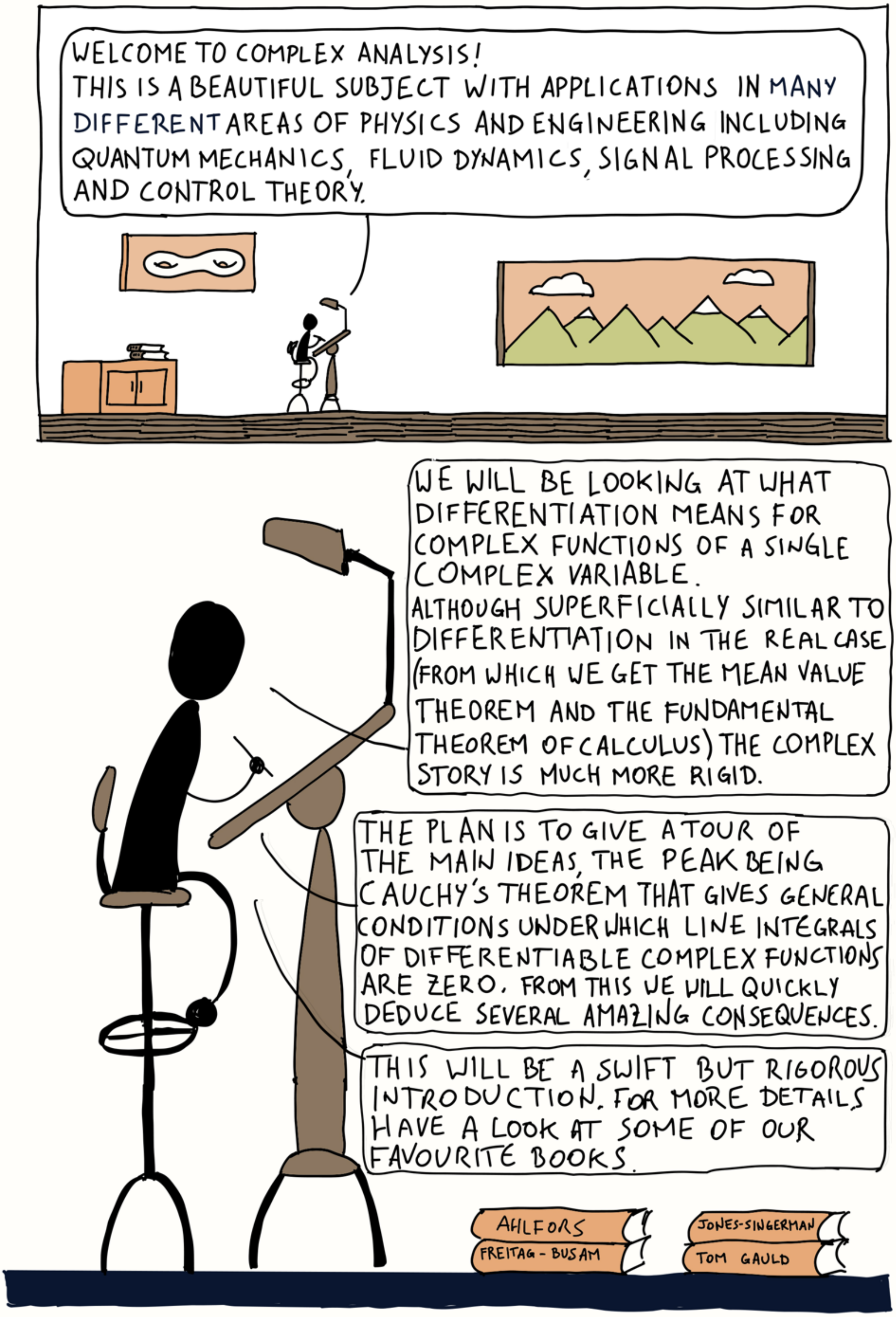
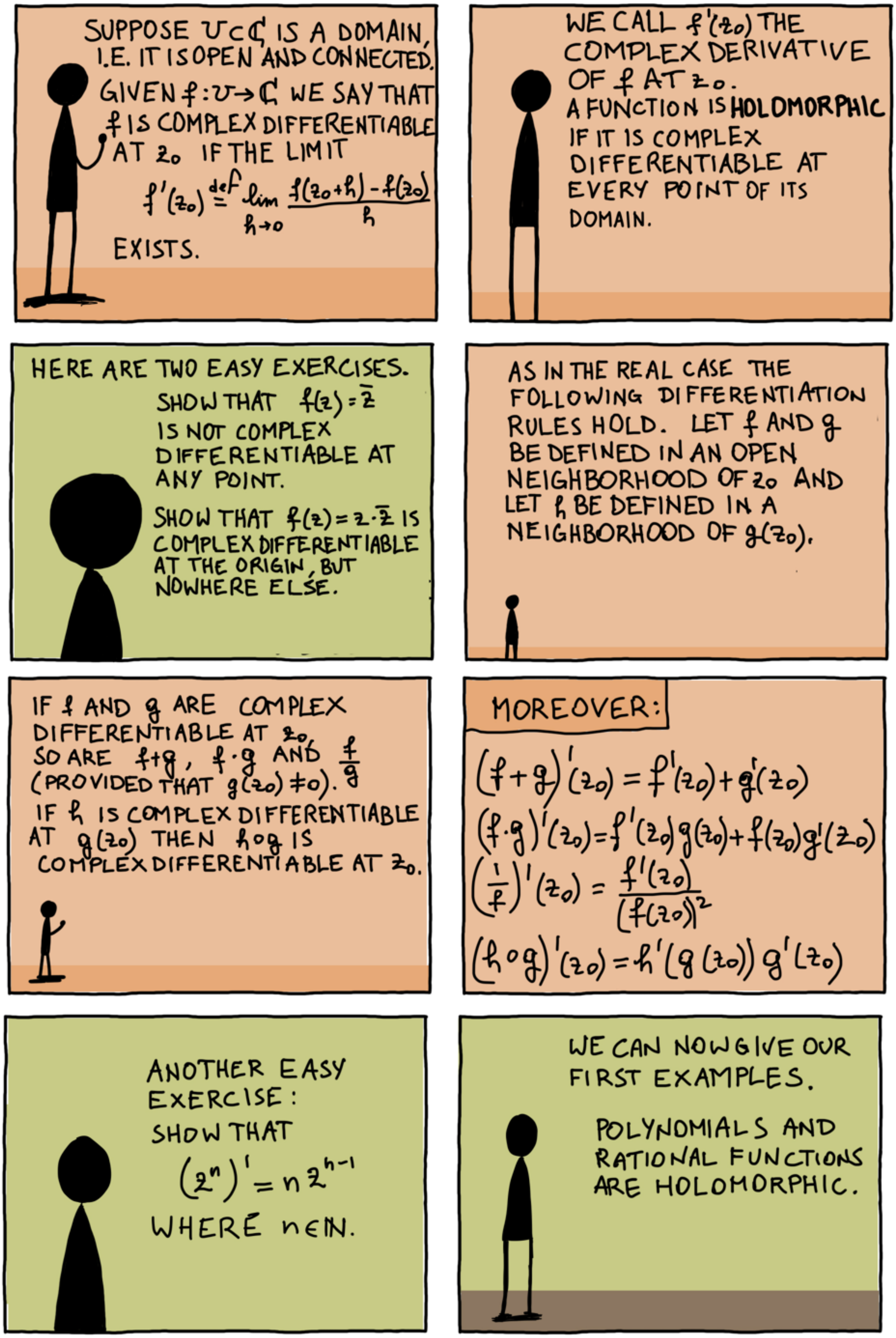
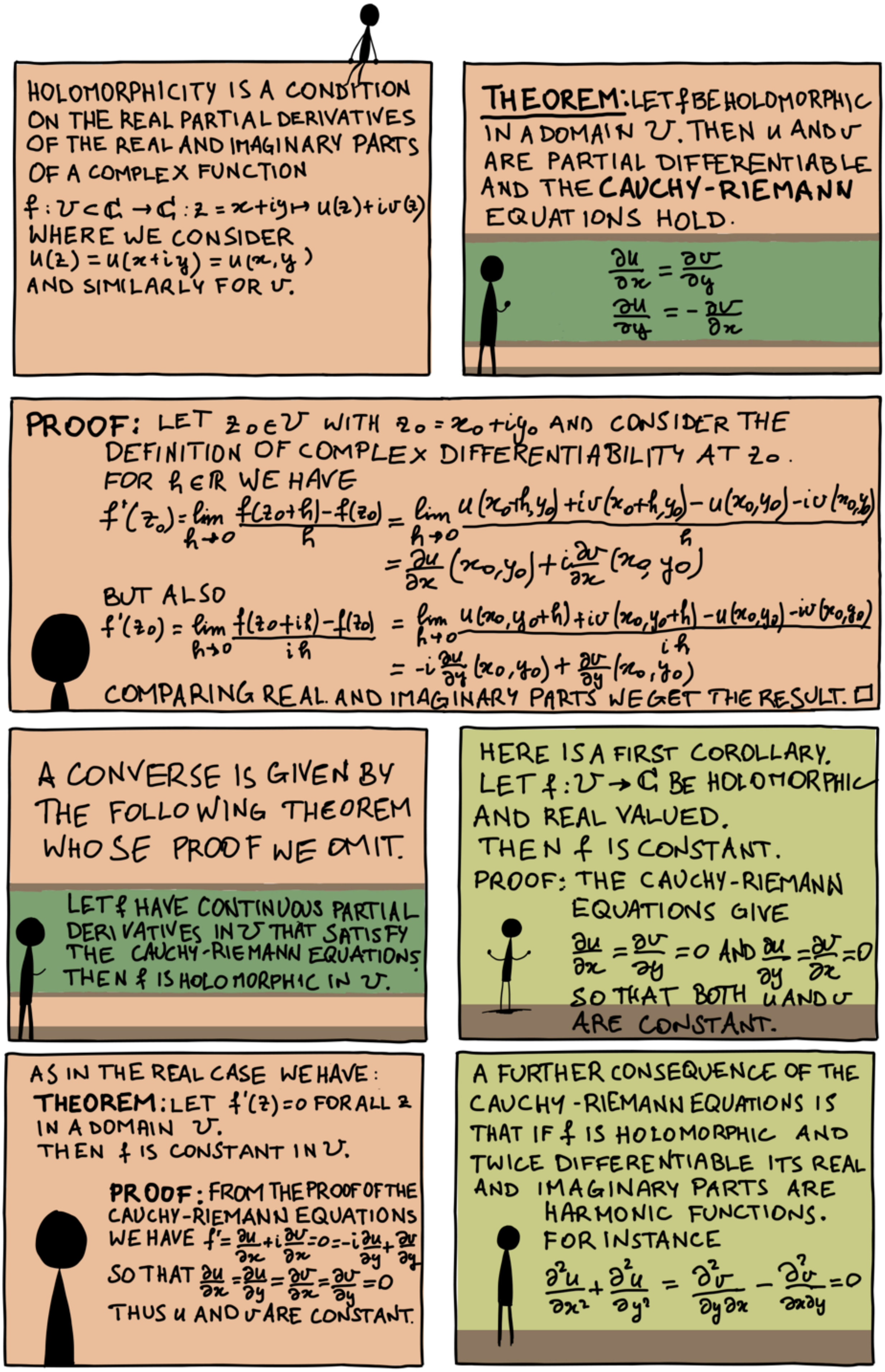
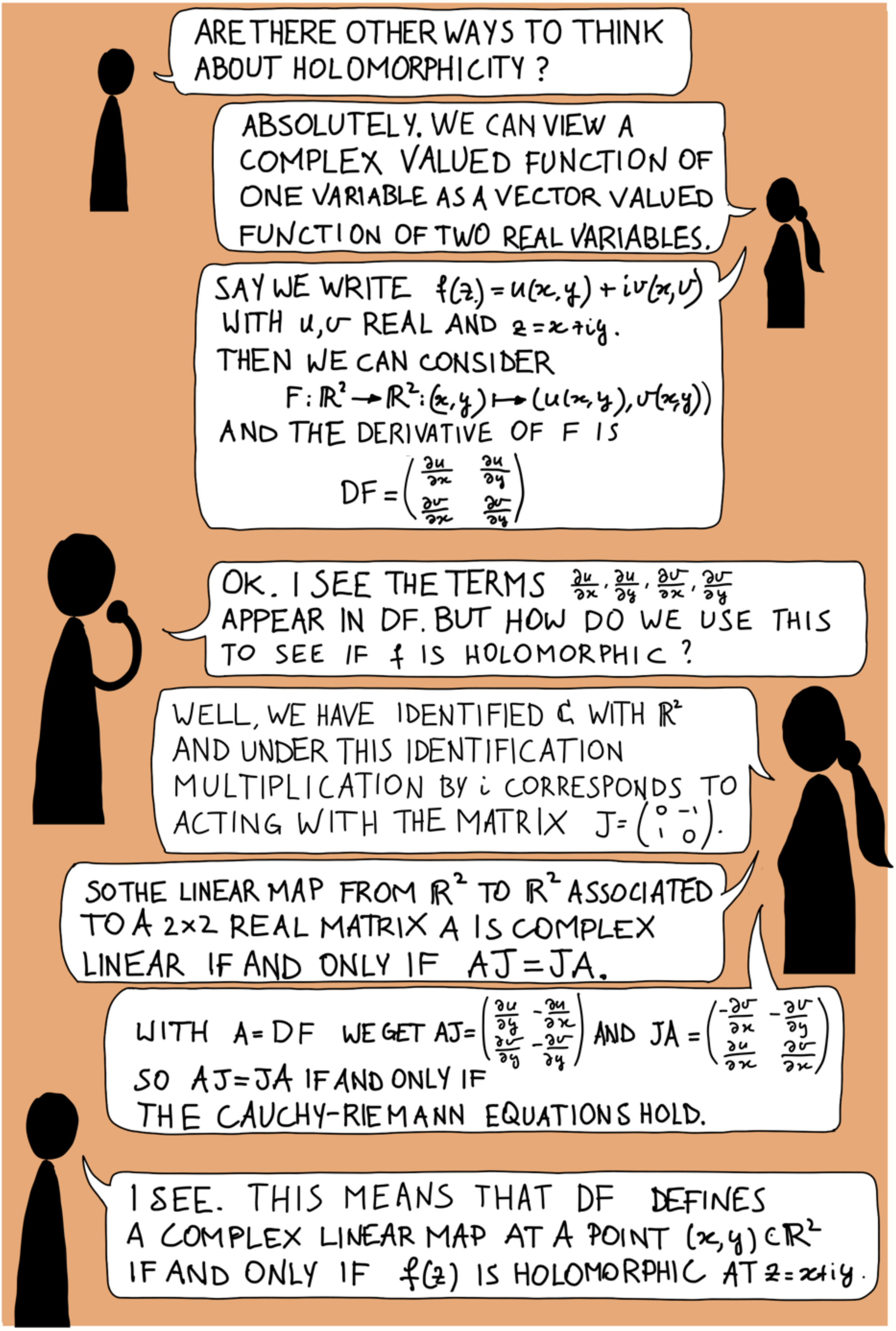



WELCOME TO COMPLEX ANALYSIS!
THIS IS A BEAUTIFUL SUBJECT WITH APPLICATIONS IN MANY DIFFERENT AREAS OF PHYSICS AND ENGINEERING INCLUDING QUANTUM MECHANICS, FLUID DYNAMICS, SIGNAL PROCESSING AND CONTROL THEORY.
WE WILL BE LOOKING AT WHAT DIFFERENTIATION MEANS FOR COMPLEX FUNCTIONS OF A SINGLE COMPLEX VARIABLE.
ALTHOUGH SUPERFICIALLY SIMILAR TO DIFFERENTIATION IN THE REAL CASE (FROM WHICH WE GET THE MEAN VALUE THEOREM AND THE FUNDAMENTAL THEOREM OF CALCULUS) THE COMPLEX STORY IS MUCH MORE RIGID.
THE PLAN IS TO GIVE A TOUR OF THE MAIN IDEAS, THE PEAK BEING CAUCHY'S THEOREM THAT GIVES GENERAL CONDITIONS UNDER WHICH LINE INTEGRALS OF DIFFERENTIABLE COMPLEX FUNCTIONS ARE ZERO. FROM THIS WE WILL QUICKLY DEDUCE SEVERAL AMAZING CONSEQUENCES.
THIS WILL BE A SWIFT BUT RIGOROUS INTRODUCTION. FOR MORE DETAILS HAVE A LOOK AT SOME OF OUR FAVOURITE BOOKS.
AHLFORS
FREITAG - BUSAM
JONES-SINGERMAN
TOM GAULD


SUPPOSE $U \subset \mathbb{C}$ IS A DOMAIN, I.E. IT IS OPEN AND CONNECTED. GIVEN $f: U \to \mathbb{C}$ WE SAY THAT f IS COMPLEX DIFFERENTIABLE AT z_0 IF THE LIMIT
$f'(z_0) \stackrel{def}{=} \lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$
EXISTS.
WE CALL $f'(z_0)$ THE COMPLEX DERIVATIVE OF f AT z_0.
A FUNCTION IS **HOLOMORPHIC** IF IT IS COMPLEX DIFFERENTIABLE AT EVERY POINT OF ITS DOMAIN.
HERE ARE TWO EASY EXERCISES.
SHOW THAT $f(z)=\bar{z}$ IS NOT COMPLEX DIFFERENTIABLE AT ANY POINT.
SHOW THAT $f(z)=z\cdot\bar{z}$ IS COMPLEX DIFFERENTIABLE AT THE ORIGIN, BUT NOWHERE ELSE.
AS IN THE REAL CASE THE FOLLOWING DIFFERENTIATION RULES HOLD. LET f AND g BE DEFINED IN AN OPEN NEIGHBORHOOD OF z_0 AND LET h BE DEFINED IN A NEIGHBORHOOD OF $g(z_0)$.
IF f AND g ARE COMPLEX DIFFERENTIABLE AT z_0, SO ARE $f+g$, $f\cdot g$ AND $\frac{f}{g}$ (PROVIDED THAT $g(z_0) \neq 0$).
IF h IS COMPLEX DIFFERENTIABLE AT $g(z_0)$ THEN $h \circ g$ IS COMPLEX DIFFERENTIABLE AT z_0.
MOREOVER:
$(f+g)'(z_0) = f'(z_0)+g'(z_0)$
$(f\cdot g)'(z_0)=f'(z_0)g(z_0)+f(z_0)g'(z_0)$
$\left(\frac{1}{f}\right)'(z_0) = \frac{f'(z_0)}{(f(z_0))^2}$
$(h \circ g)'(z_0) = h'(g(z_0))\, g'(z_0)$
ANOTHER EASY EXERCISE:
SHOW THAT $(z^n)' = n z^{n-1}$
WHERE $n \in \mathbb{N}$.
WE CAN NOW GIVE OUR FIRST EXAMPLES.
POLYNOMIALS AND RATIONAL FUNCTIONS ARE HOLOMORPHIC.


HOLOMORPHICITY IS A CONDITION ON THE REAL PARTIAL DERIVATIVES OF THE REAL AND IMAGINARY PARTS OF A COMPLEX FUNCTION
$f: \mathcal{U} \subset \mathbb{C} \to \mathbb{C}: z = x+iy \mapsto u(z)+iv(z)$
WHERE WE CONSIDER
$u(z) = u(x+iy) = u(x,y)$
AND SIMILARLY FOR v.
THEOREM: LET f BE HOLOMORPHIC IN A DOMAIN $\mathcal{U}$. THEN u AND v ARE PARTIAL DIFFERENTIABLE AND THE CAUCHY-RIEMANN EQUATIONS HOLD.
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$
$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
PROOF: LET $z_0 \in \mathcal{U}$ WITH $z_0 = x_0 + iy_0$ AND CONSIDER THE DEFINITION OF COMPLEX DIFFERENTIABILITY AT z_0.
FOR $h \in \mathbb{R}$ WE HAVE
$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h \to 0} \frac{u(x_0+h,y_0)+iv(x_0+h,y_0)-u(x_0,y_0)-iv(x_0,y_0)}{h}$
$= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$
BUT ALSO
$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+ih)-f(z_0)}{ih} = \lim_{h \to 0} \frac{u(x_0,y_0+h)+iv(x_0,y_0+h)-u(x_0,y_0)-iv(x_0,y_0)}{ih}$
$= -i\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0)$
COMPARING REAL AND IMAGINARY PARTS WE GET THE RESULT. □
A CONVERSE IS GIVEN BY THE FOLLOWING THEOREM WHOSE PROOF WE OMIT.
LET f HAVE CONTINUOUS PARTIAL DERIVATIVES IN $\mathcal{U}$ THAT SATISFY THE CAUCHY-RIEMANN EQUATIONS. THEN f IS HOLOMORPHIC IN $\mathcal{U}$.
HERE IS A FIRST COROLLARY.
LET $f: \mathcal{U} \to \mathbb{C}$ BE HOLOMORPHIC AND REAL VALUED.
THEN f IS CONSTANT.
PROOF: THE CAUCHY-RIEMANN EQUATIONS GIVE
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$ AND $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0$
SO THAT BOTH u AND v ARE CONSTANT.
AS IN THE REAL CASE WE HAVE:
THEOREM: LET $f'(z)=0$ FOR ALL z IN A DOMAIN $\mathcal{U}$.
THEN f IS CONSTANT IN $\mathcal{U}$.
PROOF: FROM THE PROOF OF THE CAUCHY-RIEMANN EQUATIONS WE HAVE $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$
SO THAT $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$
THUS u AND v ARE CONSTANT.
A FURTHER CONSEQUENCE OF THE CAUCHY-RIEMANN EQUATIONS IS THAT IF f IS HOLOMORPHIC AND TWICE DIFFERENTIABLE ITS REAL AND IMAGINARY PARTS ARE HARMONIC FUNCTIONS.
FOR INSTANCE
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0$


ARE THERE OTHER WAYS TO THINK ABOUT HOLOMORPHICITY?
ABSOLUTELY. WE CAN VIEW A COMPLEX VALUED FUNCTION OF ONE VARIABLE AS A VECTOR VALUED FUNCTION OF TWO REAL VARIABLES.
SAY WE WRITE $f(z) = u(x,y) + iv(x,y)$ WITH u, v REAL AND $z = x+iy$. THEN WE CAN CONSIDER $F: \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto (u(x,y), v(x,y))$ AND THE DERIVATIVE OF F IS $DF = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$
OK. I SEE THE TERMS $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ APPEAR IN DF. BUT HOW DO WE USE THIS TO SEE IF f IS HOLOMORPHIC?
WELL, WE HAVE IDENTIFIED $\mathbb{C}$ WITH $\mathbb{R}^2$ AND UNDER THIS IDENTIFICATION MULTIPLICATION BY i CORRESPONDS TO ACTING WITH THE MATRIX $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
SO THE LINEAR MAP FROM $\mathbb{R}^2$ TO $\mathbb{R}^2$ ASSOCIATED TO A 2×2 REAL MATRIX A IS COMPLEX LINEAR IF AND ONLY IF $AJ = JA$.
WITH $A = DF$ WE GET $AJ = \begin{pmatrix} \frac{\partial u}{\partial y} & -\frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial y} \end{pmatrix}$ AND $JA = \begin{pmatrix} -\frac{\partial v}{\partial x} & -\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \end{pmatrix}$ SO $AJ = JA$ IF AND ONLY IF THE CAUCHY-RIEMANN EQUATIONS HOLD.
I SEE. THIS MEANS THAT DF DEFINES A COMPLEX LINEAR MAP AT A POINT $(x,y) \in \mathbb{R}^2$ IF AND ONLY IF $f(z)$ IS HOLOMORPHIC AT $z = x+iy$.

WE NEXT DISCUSS SOME PURELY FORMAL NOTATION FOR "COMPLEX DERIVATIVES" THAT IS USEFUL IN CAPTURING HOLOMORPHICITY AND THE CAUCHY-RIEMANN EQUATIONS.

DEFINITION: FOR $z = x+iy$

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

LEMMA: LET $f: U \to \mathbb{C}$ IN $\mathcal{C}^1$ BE $f(z) = u(x,y) + iv(x,y)$ WITH $z = x+iy$

THEN

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

PROOF:

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y}\right)$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

EXAMPLE:

$$\frac{\partial}{\partial z}(z^2) = 2z$$

PROOF:

$$\frac{\partial}{\partial z}(z^2) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x+iy)^2$$

$$= \frac{1}{2}2(x+iy) - \frac{i}{2}2(x+iy)\cdot i$$

$$= 2(x+iy)$$

FACTS: IF $f, g \in \mathcal{C}^1$

$$\frac{\partial}{\partial \bar{z}}(f+g) = \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}$$

$$\frac{\partial}{\partial \bar{z}}(fg) = \frac{\partial f}{\partial \bar{z}}g + f\frac{\partial g}{\partial \bar{z}}$$

COROLLARY: IF $f \in \mathcal{C}^2$ THEN

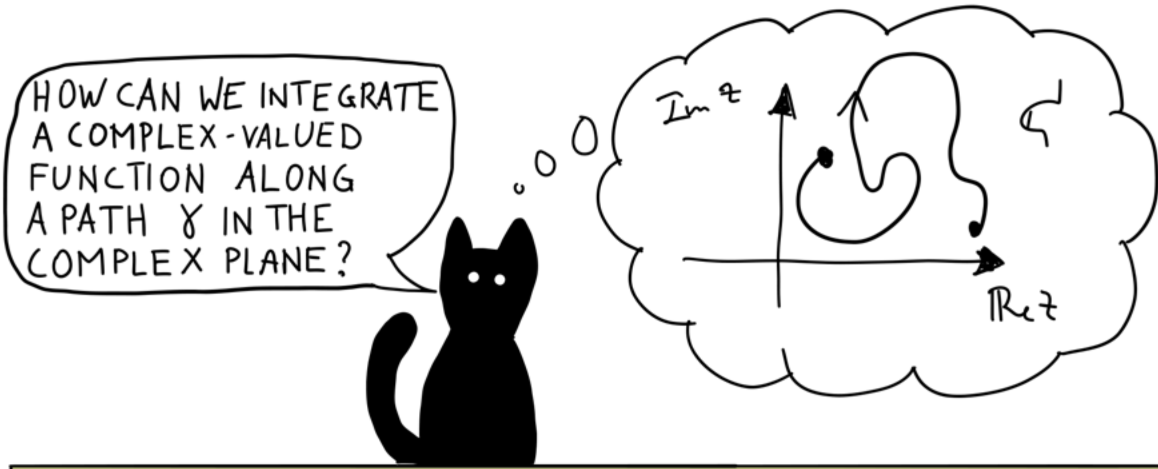
$\frac{\partial f}{\partial \bar{z}} = 0 \iff f$ IS HOLOMORPHIC

PROOF:

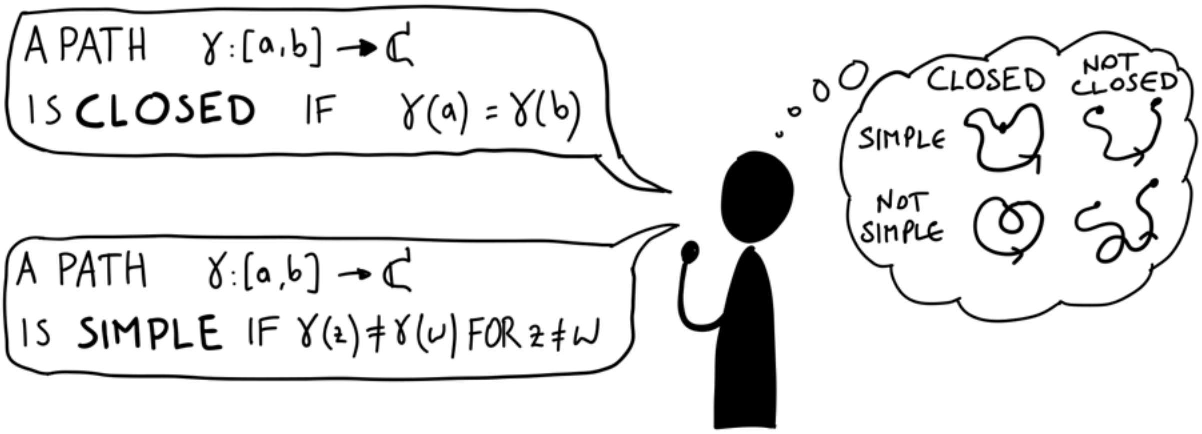
$$\frac{\partial f}{\partial \bar{z}} \iff \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \text{ AND } \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

AND THESE ARE THE CAUCHY-RIEMANN EQUATIONS FOR f □

EXERCISE:
USE WHAT IS ON THIS PAGE TO SHOW THAT IF f AND g ARE HOLOMORPHIC SO ARE $f+g$ AND $f \cdot g$



DEFINITION: A **PATH** IN $\mathbb{C}$ IS A SMOOTH $\gamma : [a,b] \to \mathbb{C}$.



DEFINITION: ASSUME $F : U \to \mathbb{C}$ IS CONTINUOUS AND U IS AN OPEN NEIGHBORHOOD OF $\mathrm{Im}(\gamma)$. THE **LINE INTEGRAL** OF F OVER γ IS

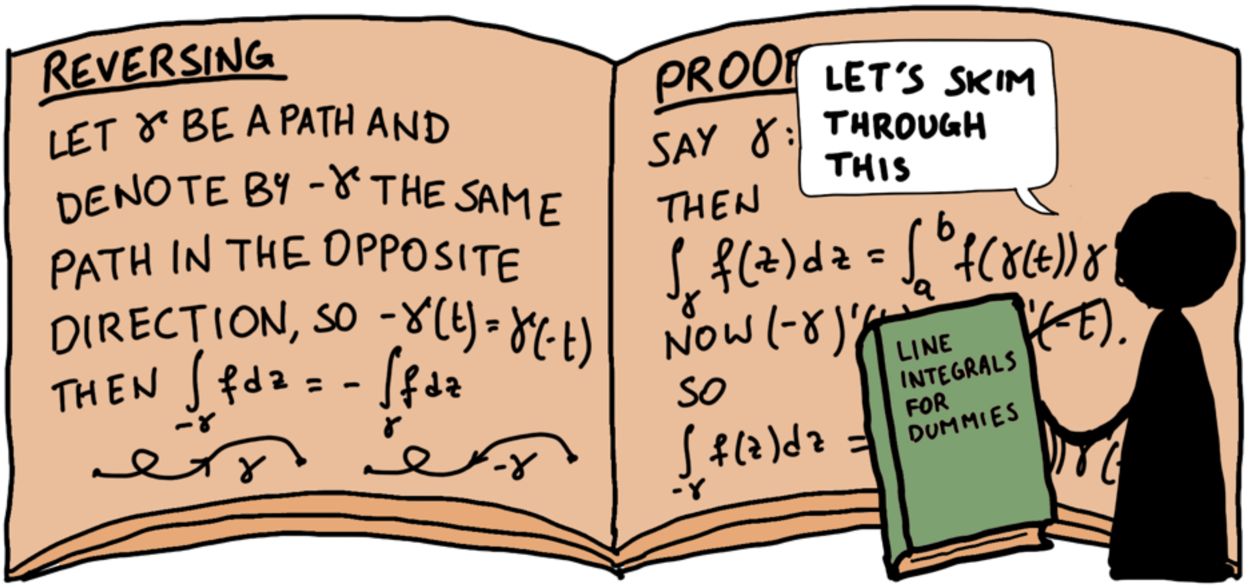
$$\int_\gamma F\,dz := \int_a^b F(\gamma(t))\,\gamma'(t)\,dt$$

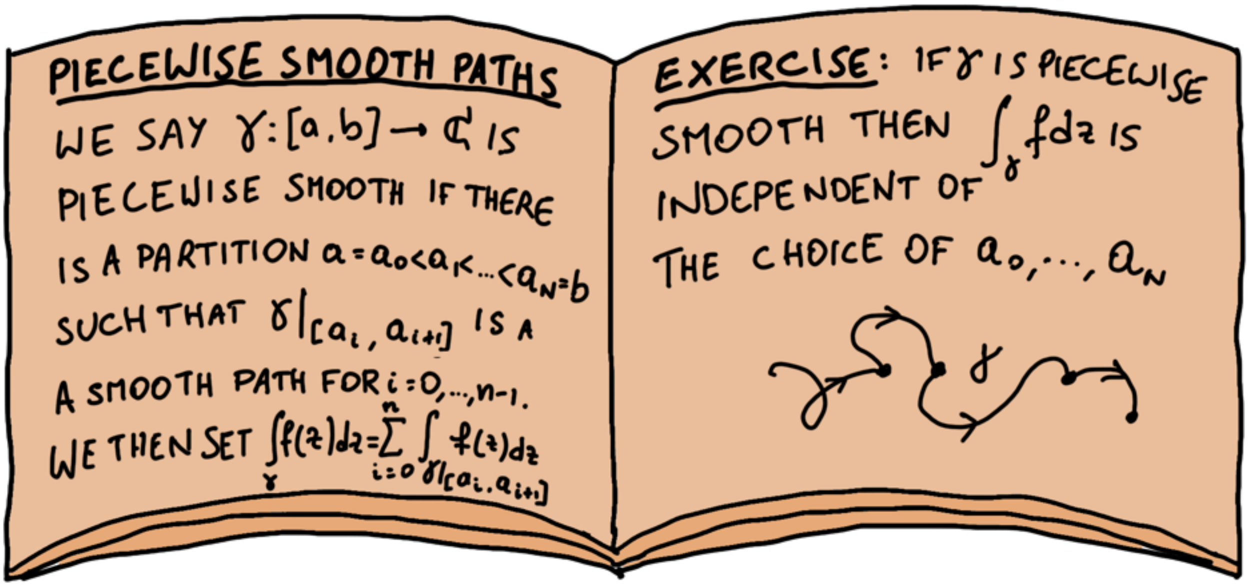
WHERE $\gamma : [a,b] \to \mathbb{C}$.

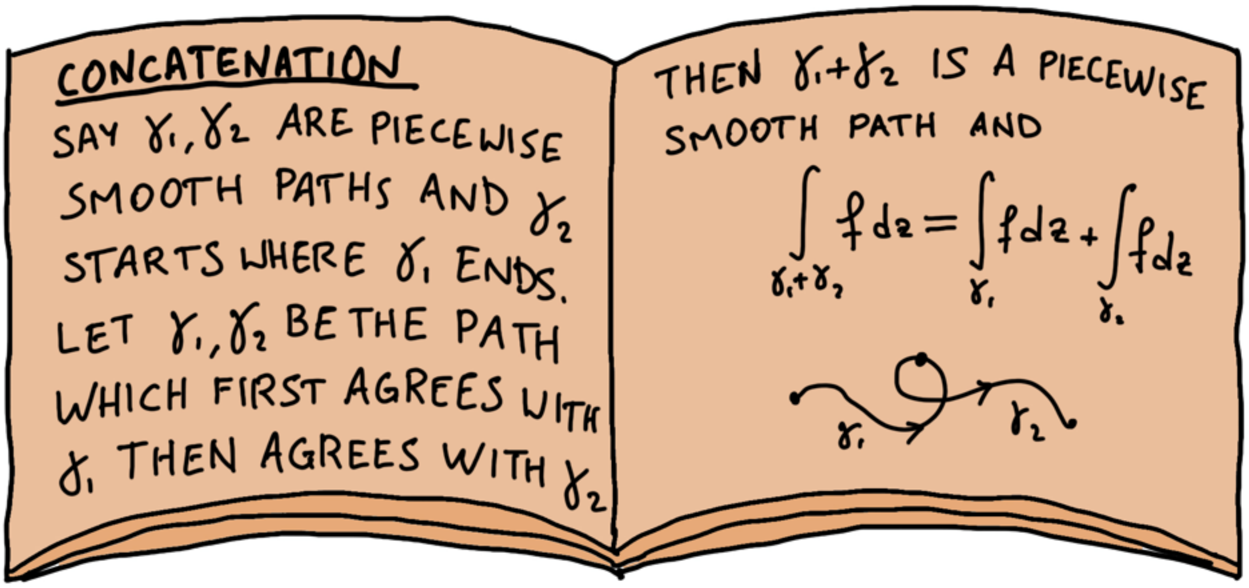
EXERCISE

LET $\gamma : [0, 2\pi] \to \mathbb{C}$ BE $\gamma(t) = e^{it}$ THEN

$$\int_\gamma z^n\,dz = \begin{cases} 2\pi i & \text{IF } n = -1 \\ 0 & \text{IF } n \neq -1 \end{cases}$$


REVERSING
LET γ BE A PATH AND DENOTE BY $-\gamma$ THE SAME PATH IN THE OPPOSITE DIRECTION, SO $-\gamma(t) = \gamma(-t)$
THEN $\int_{-\gamma} f\,dz = -\int_{\gamma} f\,dz$
γ
$-\gamma$
PROOF
SAY γ:
THEN
$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma$
NOW $(-\gamma)'($
$(-t)$.
SO
$\int_{-\gamma} f(z)\,dz =$
LET'S SKIM THROUGH THIS
LINE INTEGRALS FOR DUMMIES


PIECEWISE SMOOTH PATHS
WE SAY $\gamma : [a,b] \to \mathbb{C}$ IS PIECEWISE SMOOTH IF THERE IS A PARTITION $a = a_0 < a_1 < \dots < a_N = b$ SUCH THAT $\gamma|_{[a_i, a_{i+1}]}$ IS A A SMOOTH PATH FOR $i = 0, \dots, n-1$.
WE THEN SET $\int_\gamma f(z)\,dz = \sum_{i=0}^{n} \int_{\gamma|[a_i, a_{i+1}]} f(z)\,dz$
EXERCISE: IF γ IS PIECEWISE SMOOTH THEN $\int_\gamma f\,dz$ IS INDEPENDENT OF THE CHOICE OF $a_0, \dots, a_N$
γ


CONCATENATION
SAY γ_1, γ_2 ARE PIECEWISE SMOOTH PATHS AND γ_2 STARTS WHERE γ_1 ENDS.
LET γ_1, γ_2 BE THE PATH WHICH FIRST AGREES WITH γ_1 THEN AGREES WITH γ_2
THEN $\gamma_1 + \gamma_2$ IS A PIECEWISE SMOOTH PATH AND
$\int_{\gamma_1+\gamma_2} f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz$
γ_1
γ_2

LINE INTEGRALS THROUGH PRIMITIVES

SUPPOSE f IS CONTINUOUS ON AN OPEN SET $\mathcal{U}$ AND THERE IS A SMOOTH FUNCTION g SUCH THAT $g' = f$

THEN FOR ANY PIECEWISE SMOOTH PATH $\gamma : [a,b] \to \mathbb{C}$

$$\int_\gamma f(z)\,dz = g(\gamma(b)) - g(\gamma(a))$$

REPARAMETRIZATIONS

A PATH γ IS DEFINED TO BE A MAP $\gamma : I \to \mathbb{C}$ AND NOT JUST A SUBSET OF $\mathbb{C}$. IF $\sigma : \tilde{I} \to I$ IS A DIFFEOMORPHISM WE CALL

$$\tilde{\gamma} = \gamma \circ \sigma : \tilde{I} \to \mathbb{C}$$

A REPARAMETRIZATION OF γ AND IN THIS CASE

$$\int_\gamma f(z)\,dz = \int_{\tilde{\gamma}} f(z)\,dz$$

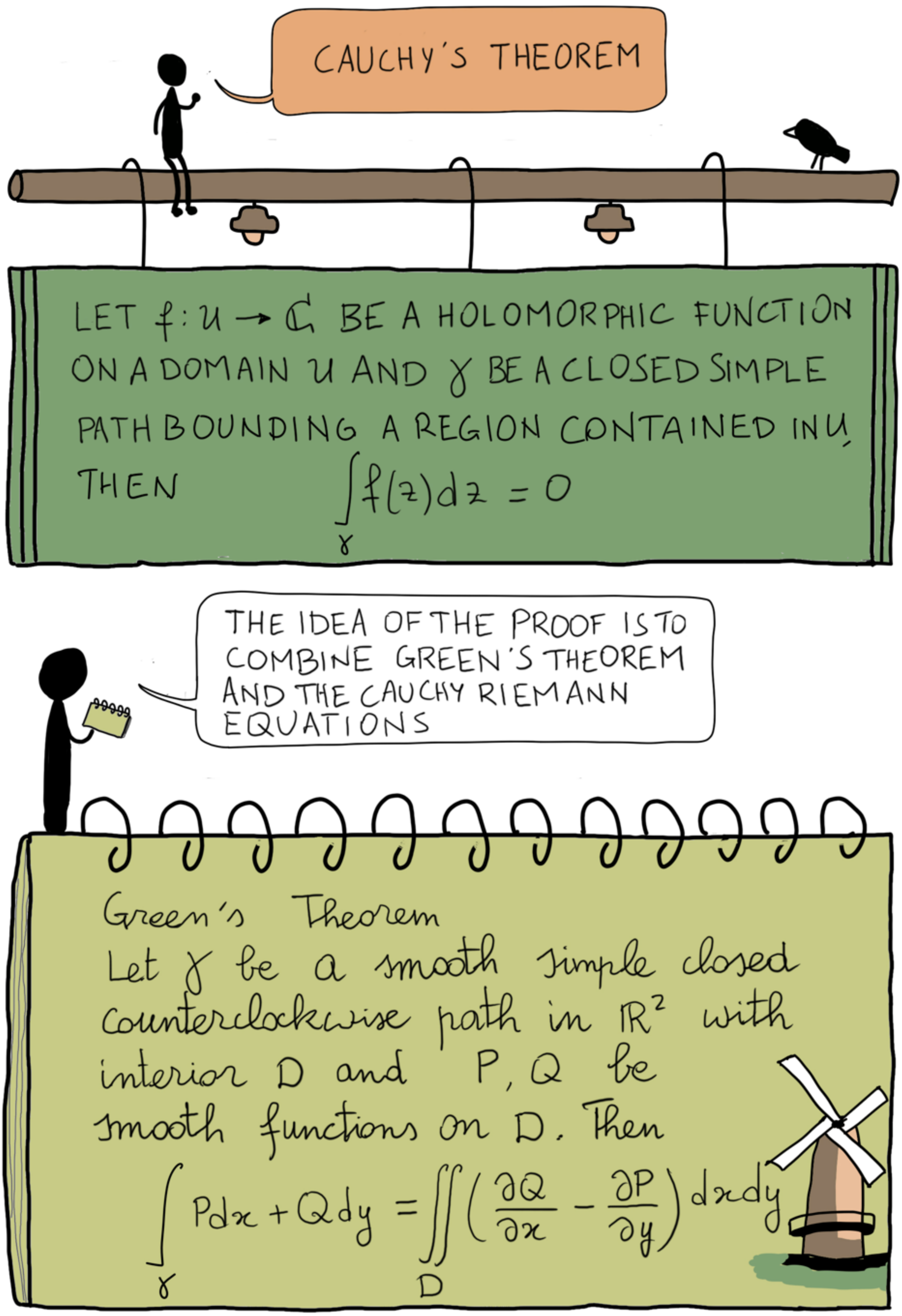
TRIANGLE INEQUALITY

WE HAVE

$$\left| \int_\gamma f(z)\,dz \right| \leq \int_\gamma |f(z)|\,|dz|$$

WHERE IF $\gamma : I \to \mathbb{C}$ THE RIGHT-HAND SIDE MEANS

$$\int_\gamma |f(z)|\,|dz| = \int_I |f(\gamma(t))|\,|\gamma'(t)|\,dt$$


CAUCHY'S THEOREM
LET $f: \mathcal{U} \to \mathbb{C}$ BE A HOLOMORPHIC FUNCTION ON A DOMAIN $\mathcal{U}$ AND γ BE A CLOSED SIMPLE PATH BOUNDING A REGION CONTAINED IN $\mathcal{U}$. THEN $\int_\gamma f(z)\,dz = 0$
THE IDEA OF THE PROOF IS TO COMBINE GREEN'S THEOREM AND THE CAUCHY RIEMANN EQUATIONS
Green's Theorem
Let γ be a smooth simple closed counterclockwise path in $\mathbb{R}^2$ with interior D and P, Q be smooth functions on D. Then
$\int_\gamma P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

PROOF OF CAUCHY'S THEOREM

WRITE

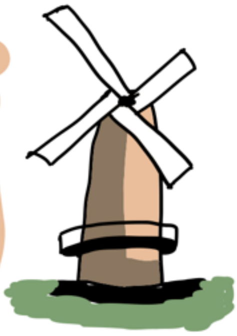
$$f(z) = u(x,y) + i v(x,y) \quad \text{WHERE } z = x+iy$$

THEN

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx + i\,dy)$$

$$= \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (v\,dx + u\,dy)$$

NOW APPLY GREEN'S THEOREM. TWICE.



WE HAVE

$$\int_\gamma (u\,dx - v\,dy) = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

AND WE HAVE

$$\int_\gamma (v\,dx + u\,dy) = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

RECALL:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

f IS HOLOMORPHIC, SO THE CAUCHY-RIEMANN EQUATIONS HOLD

RECALL:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

SO

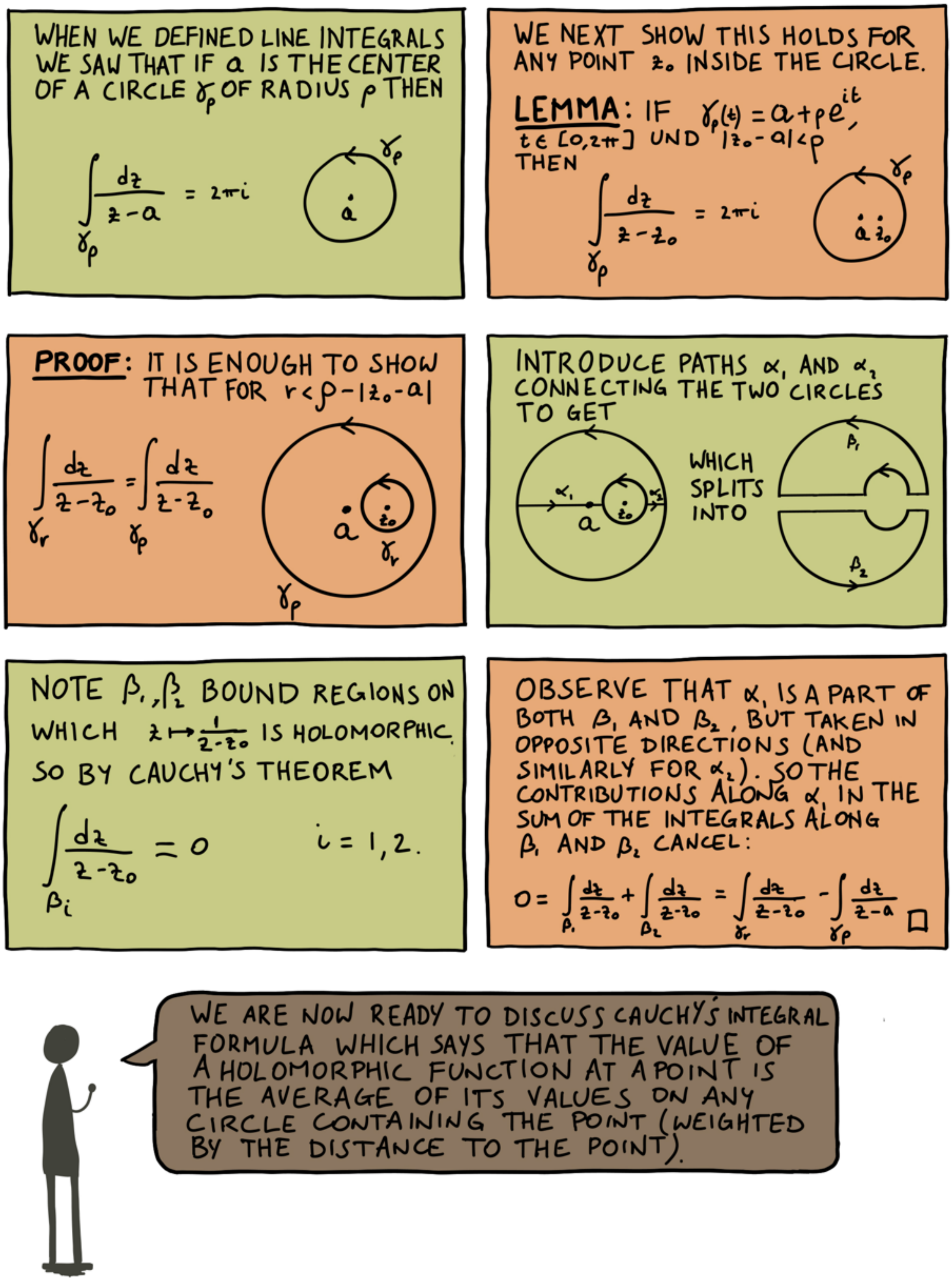
$$\int_\gamma (u\,dx - v\,dx) = 0$$

SO

$$\int_\gamma (u\,dy + v\,dy) = 0$$

WHICH TOGETHER IMPLY

$$\int_\gamma f(z)\,dz = 0$$


WHEN WE DEFINED LINE INTEGRALS WE SAW THAT IF a IS THE CENTER OF A CIRCLE γ_ρ OF RADIUS ρ THEN
$\int_{\gamma_\rho} \frac{dz}{z-a} = 2\pi i$
γ_ρ
a
WE NEXT SHOW THIS HOLDS FOR ANY POINT z_0 INSIDE THE CIRCLE.
LEMMA: IF $\gamma_\rho(t) = a + \rho e^{it}$, $t \in [0, 2\pi]$ UND $|z_0 - a| < \rho$ THEN
$\int_{\gamma_\rho} \frac{dz}{z-z_0} = 2\pi i$
γ_ρ
a z_0
PROOF: IT IS ENOUGH TO SHOW THAT FOR $r < \rho - |z_0 - a|$
$\int_{\gamma_r} \frac{dz}{z-z_0} = \int_{\gamma_\rho} \frac{dz}{z-z_0}$
a z_0 γ_r γ_ρ
INTRODUCE PATHS α_1 AND α_2 CONNECTING THE TWO CIRCLES TO GET
α_1 α_2 a z_0
WHICH SPLITS INTO
β_1
β_2
NOTE β_1, β_2 BOUND REGIONS ON WHICH $z \mapsto \frac{1}{z-z_0}$ IS HOLOMORPHIC. SO BY CAUCHY'S THEOREM
$\int_{\beta_i} \frac{dz}{z-z_0} = 0 \qquad i = 1, 2.$
OBSERVE THAT α_1 IS A PART OF BOTH β_1 AND β_2, BUT TAKEN IN OPPOSITE DIRECTIONS (AND SIMILARLY FOR α_2). SO THE CONTRIBUTIONS ALONG α_1 IN THE SUM OF THE INTEGRALS ALONG β_1 AND β_2 CANCEL:
$0 = \int_{\beta_1} \frac{dz}{z-z_0} + \int_{\beta_2} \frac{dz}{z-z_0} = \int_{\gamma_r} \frac{dz}{z-z_0} - \int_{\gamma_\rho} \frac{dz}{z-a}$ □
WE ARE NOW READY TO DISCUSS CAUCHY'S INTEGRAL FORMULA WHICH SAYS THAT THE VALUE OF A HOLOMORPHIC FUNCTION AT A POINT IS THE AVERAGE OF ITS VALUES ON ANY CIRCLE CONTAINING THE POINT (WEIGHTED BY THE DISTANCE TO THE POINT).

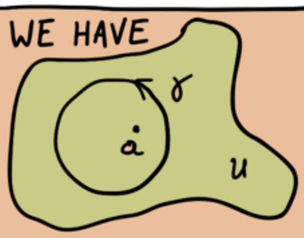
CAUCHY'S INTEGRAL FORMULA

LET $f : U \to \mathbb{C}$ BE HOLOMORPHIC AND $a \in U$ BE INSIDE A CIRCLE γ CONTAINED IN U. THEN

$$f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\,dz$$

PROOF

WE HAVE



DEFINE

$$g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$$

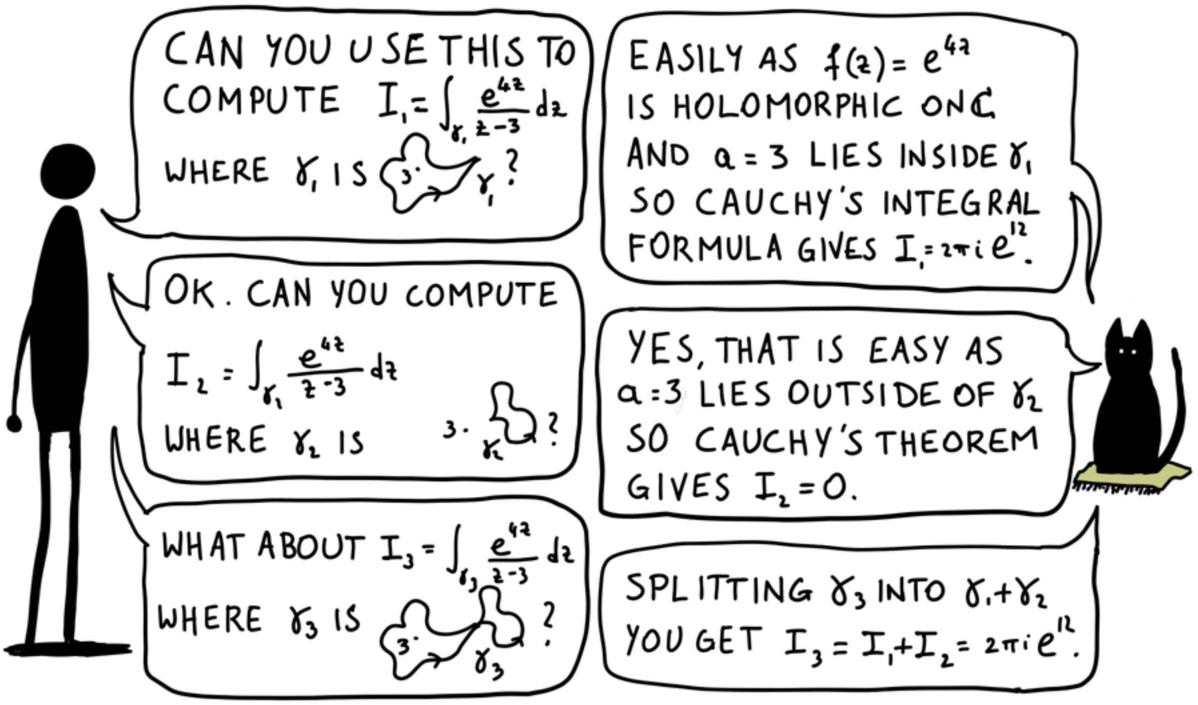
APPLY CAUCHY'S THEOREM TO g ALONG γ

TO GET

$$0 = \int_\gamma g(z)\,dz = \int_\gamma \frac{f(z)}{z-a}\,dz - \int_\gamma \frac{f(a)}{z-a}\,dz$$

SO

$$\int_\gamma \frac{f(z)}{z-a}\,dz = f(a) \cdot \int_\gamma \frac{dz}{z-a} = 2\pi i f(a) \qquad \square$$



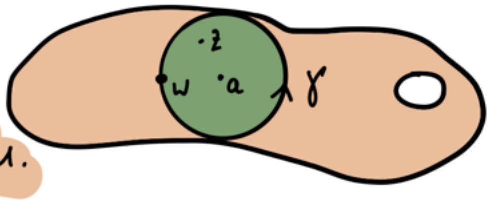
HOLOMORPHIC FUNCTIONS ARE ANALITIC

LET $f : U \to \mathbb{C}$ BE HOLOMORPHIC. THEN f IS ANALYTIC ON U, THAT IS, IF $a \in U$ THERE IS A DISC D CENTRED AT a SUCH THAT

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad \text{FOR SOME } c_n \in \mathbb{C}.$$

PROOF:

LET D BE A DISC CENTRED AT a AND CONTAINED IN U.



AND LET γ BE THE CIRCULAR CONTOUR AROUND D.

CAUCHY'S INTEGRAL FORMULA SAYS

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} \, dw$$

AND USING THE GEOMETRIC SERIES WE GET

$$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{w-a} \cdot \frac{1}{1-\left(\frac{z-a}{w-a}\right)} = \frac{1}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n$$

PUTTING THESE TOGETHER GIVES

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-a} \, dw \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a}\right)^n = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}} \, dw \, (z-a)^n$$

SO WE SET

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-a)^{n+1}}$$

AND WE ARE DONE.

THE IDENTITY THEOREM

THIS IS THE TECHNICAL VERSION

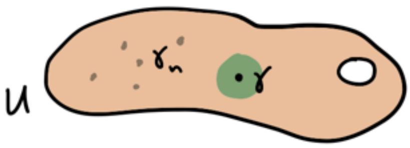
THIS IS THE AMAZING CONSEQUENCE

LET $f: U \to \mathbb{C}$ BE HOLOMORPHIC WHERE $U \subset \mathbb{C}$ IS OPEN AND CONNECTED. SUPPOSE $(c_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ IS A SEQUENCE WITH $f(c_n) = 0$ FOR ALL n. IF $(c_n)_{n\in\mathbb{N}}$ HAS AN ACCUMULATION POINT IN U THEN $f \equiv 0$ ON U.

LET $g: U \to \mathbb{C}$ AND $h: U \to \mathbb{C}$ BE HOLOMORPHIC WHERE $U \subset \mathbb{C}$ IS OPEN AND CONNECTED. IF $g = h$ ON SOME OPEN $V \subseteq U$ THEN $g \equiv h$.

LEMMA: LET $f: U \to \mathbb{C}$ BE HOLOMORPHIC. SUPPOSE $(\gamma_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ HAS AN ACCUMULATION POINT $\gamma \in U$ AND $f(\gamma_n) = 0$ FOR ALL n. THEN THERE IS AN OPEN DISC D CENTERED AT γ SUCH THAT $f \equiv 0$ ON D.

PROOF: WITHOUT LOSS OF GENERALITY WE MAY TAKE $\gamma = 0$. AS $0 \in U$ AND U IS OPEN THERE IS AN OPEN DISC $D \subset U$ CENTERED AT 0.



HOLOMORPHIC FUNCTIONS ARE ANALYTIC SO WE CAN WRITE

$f(z) = \sum_{m\geq 0} a_m z^m$ ON D FOR SOME $a_m \in \mathbb{C}$

IF f IS NOT IDENTICALLY ZERO ON D THEN THERE IS A SMALLEST $r \in \mathbb{N}$ SUCH THAT $a_r \neq 0$, SO

$$f(z) = \sum_{m \geq r} a_m z^m \text{ ON } D$$
$$= z^r (a_r + a_{r+1} z + \dots)$$

AS $a_r \neq 0$ THE TERM $(a_r + a_{r+1} z + \dots)$ IS NON ZERO FOR $|z|$ SUFFICIENTLY SMALL. SO $f(z) \neq 0$ FOR NONZERO z WITH $|z|$ SMALL.

WHICH CONTRADICTS $\gamma = 0$ BEING AN ACCUMULATION POINT OF ZEROS OF f. □

PROOF OF IDENTITY THEOREM

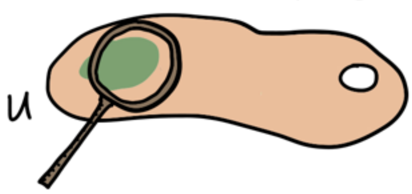
LET $G = \{c \in U : f \equiv 0 \text{ ON SOME DISC CENTERED AT } c\}$

IT IS CLEAR THAT G IS OPEN. APPLYING THE LEMMA GIVES $c \in G$ SO G IS NOT EMPTY. FINALLY IF $c \in \overline{G}$ THEN c IS THE LIMIT OF SOME SEQUENCE OF ZEROS OF f, SO THE LEMMA IMPLIES $c \in G$. SINCE U IS CONNECTED $G = U$, I.E. $f \equiv 0$ ON U.

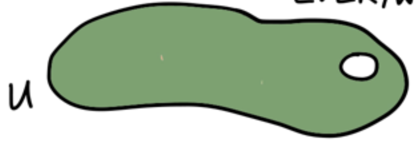
FOR THE OTHER STATEMENTS APPLY THIS TO $f = g - h$ AND USE THAT V CONTAINS A LIMIT OF POINTS $(c_n) \subseteq V$.

IN OTHER WORDS

IF I KNOW A HOLOMORPHIC FUNCTION HERE

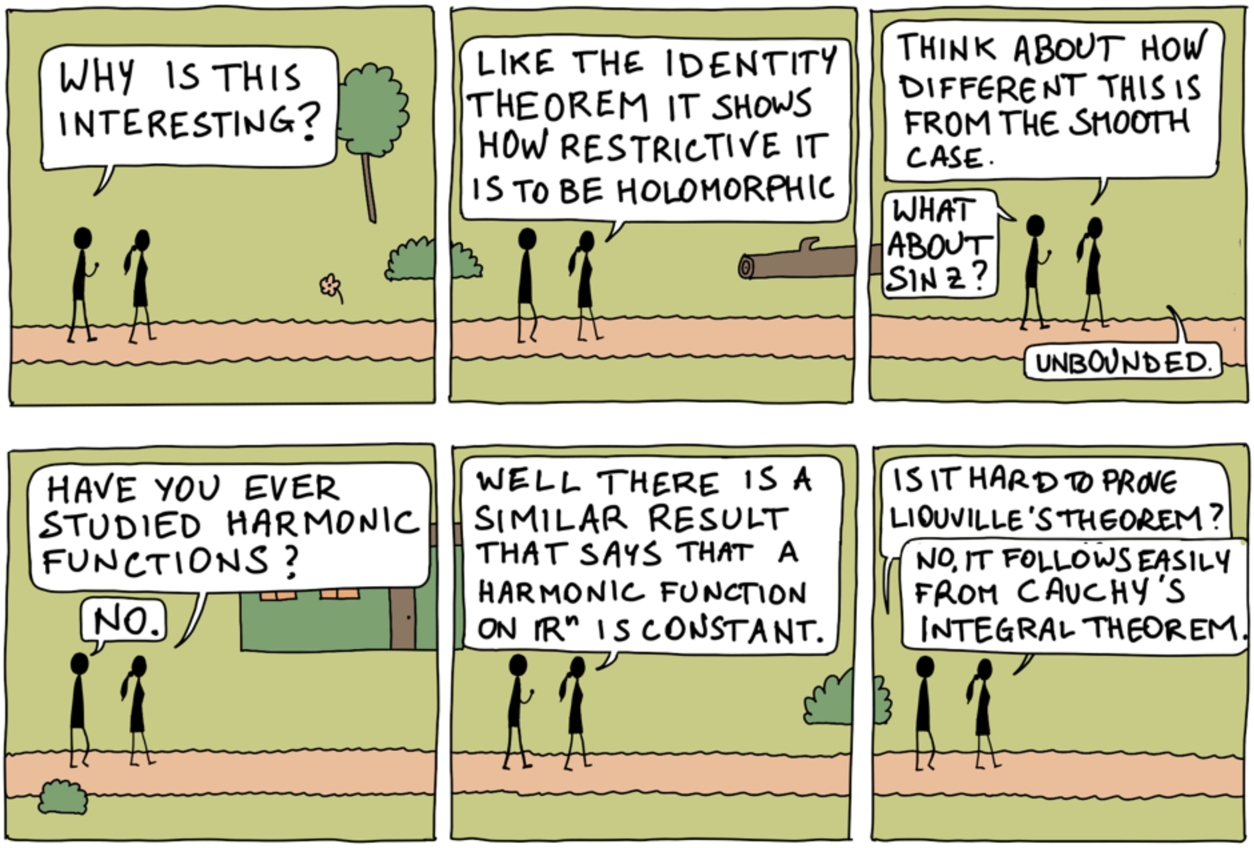


I KNOW IT EVERYWHERE



LIOUVILLE'S THEOREM

LET $f : \mathbb{C} \to \mathbb{C}$ BE HOLOMORPHIC AND BOUNDED. THEN IT IS CONSTANT.

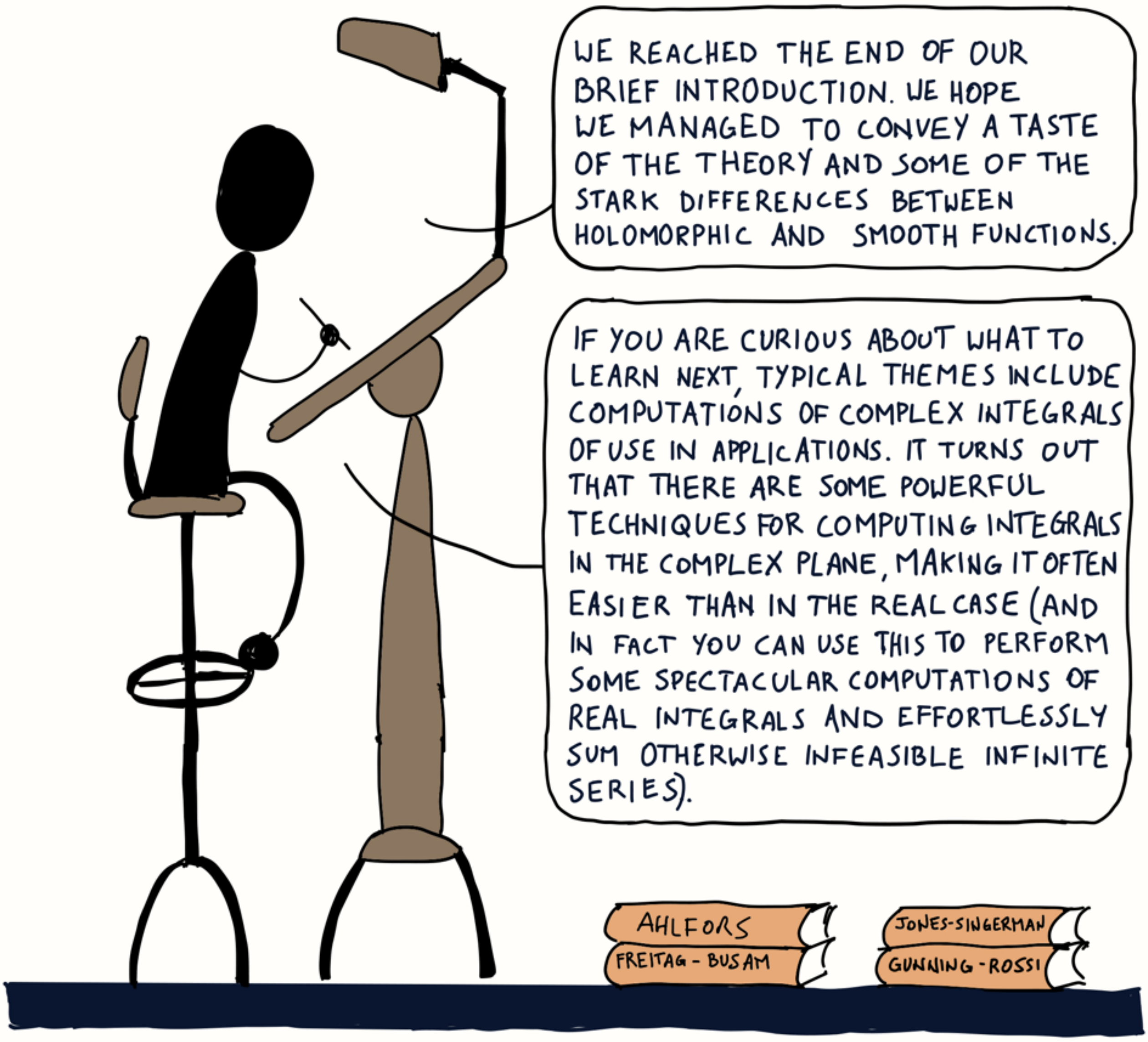


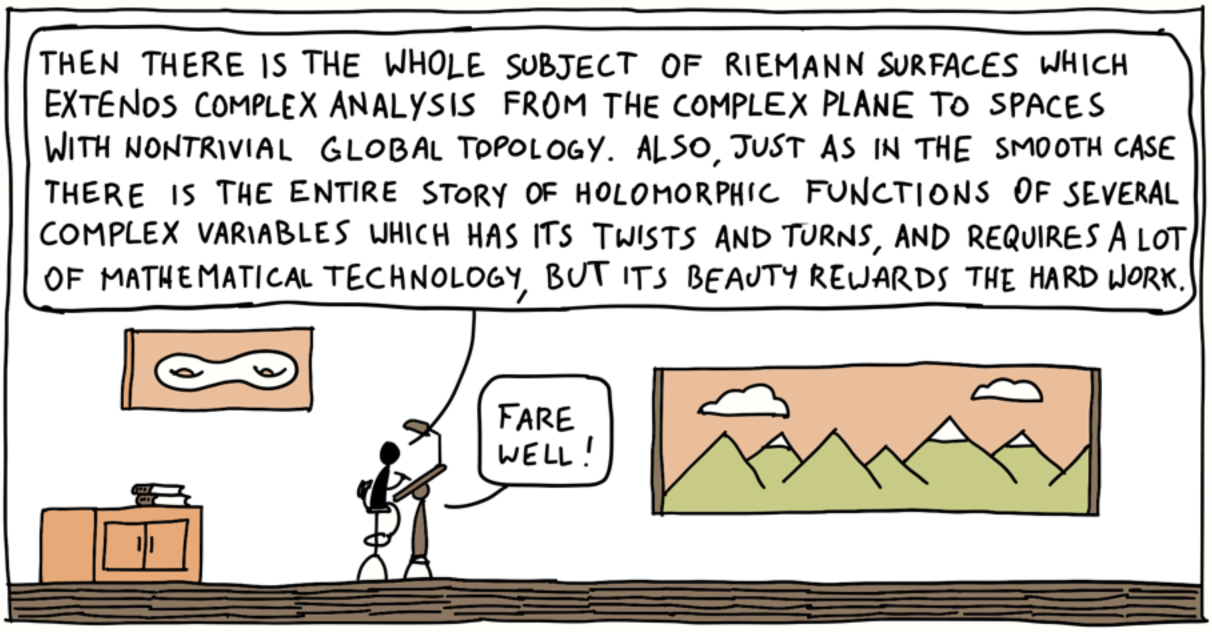
PROOF: SAY $|f(z)| \leq C$ FOR ALL z. FIX $a \in \mathbb{C}$ AND APPLY THE CAUCHY INTEGRAL FORMULA TWICE TO THE CIRCLE γ_R OF RADIUS $R > |a|$ TO GIVE

$$f(a) - f(0) = \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z-a}\,dz - \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z}\,dz = \frac{a}{2\pi i}\int_{\gamma_R} \frac{f(z)}{z(z-a)}\,dz$$

SO $$|f(a) - f(0)| \leq \frac{C|a|\,\text{length}(\gamma_R)}{2\pi R(R-|a|)} = \frac{C|a|}{R-|a|}.$$

LETTING $R \to \infty$ IMPLIES $f(a) = f(0)$. AS a WAS ARBITRARY WE CONCLUDE f IS CONSTANT.


WE REACHED THE END OF OUR BRIEF INTRODUCTION. WE HOPE WE MANAGED TO CONVEY A TASTE OF THE THEORY AND SOME OF THE STARK DIFFERENCES BETWEEN HOLOMORPHIC AND SMOOTH FUNCTIONS.
IF YOU ARE CURIOUS ABOUT WHAT TO LEARN NEXT, TYPICAL THEMES INCLUDE COMPUTATIONS OF COMPLEX INTEGRALS OF USE IN APPLICATIONS. IT TURNS OUT THAT THERE ARE SOME POWERFUL TECHNIQUES FOR COMPUTING INTEGRALS IN THE COMPLEX PLANE, MAKING IT OFTEN EASIER THAN IN THE REAL CASE (AND IN FACT YOU CAN USE THIS TO PERFORM SOME SPECTACULAR COMPUTATIONS OF REAL INTEGRALS AND EFFORTLESSLY SUM OTHERWISE INFEASIBLE INFINITE SERIES).
AHLFORS
FREITAG - BUSAM
JONES-SINGERMAN
GUNNING - ROSSI


THEN THERE IS THE WHOLE SUBJECT OF RIEMANN SURFACES WHICH EXTENDS COMPLEX ANALYSIS FROM THE COMPLEX PLANE TO SPACES WITH NONTRIVIAL GLOBAL TOPOLOGY. ALSO, JUST AS IN THE SMOOTH CASE THERE IS THE ENTIRE STORY OF HOLOMORPHIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES WHICH HAS ITS TWISTS AND TURNS, AND REQUIRES A LOT OF MATHEMATICAL TECHNOLOGY, BUT ITS BEAUTY REWARDS THE HARD WORK.
FARE WELL!

COMICS ARE A LANGUAGE THAT IS STILL LARGELY UNEXPLORED AS A MEANS OF SCHOLARLY WRITING IN THE SCIENCES. THIS SHORT BOOKLET PROPOSES AN ATTEMPT AT USING THIS LANGUAGE TO COMMUNICATE RIGOROUS MATHEMATICS, EXPLANING A CLASSICAL SUBJECT IN PURE MATHEMATICS: COMPLEX ANALYSIS. THE EXPOSITION HAS THE STANDARD LEVEL OF RIGOUR EXPECTED FOR UPPER UNDERGRADUATE MATHEMATICS STUDENTS, AND EXPLORES WAYS IN WHICH USING COMICS DIFFERS FROM STANDARD TEXT.

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